
A Theoretical and Experimental Analysis of the Acoustic Guitar

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1 Introduction and Methods

The acoustic guitar is a stringed musical instrument frequently used in popular music. Because it is common in dorm rooms across the country and can be analyzed using a simple linear model, we thought it a fitting instrument for a theoretical and experimental analysis.

In this project, we will attempt to quantify certain aspects that affect the way sound is produced by the acoustic guitar. In the remainder of this section, we will summarize the theoretical model and the signal analysis techniques we will use. In Section 2, we will cover the affects that picking location has on the harmonic distribution of a guitar note. In addition, we show that by comparing the results of the signal analysis with the theoretical model for picking location, we can approximate the picking location used to create a recorded note. In Section 3, we look into the subtle implications of tuning systems used in western music and how they influence the guitar. In Section 4, we examine natural harmonics, which, in addition to fretting, are a way to create new pitches on a guitar string.

1.1 Theoretical Analysis

Since the acoustic guitar is a stringed instrument, we plan to carry out theoretical analysis using the one dimensional wave equation. This is a completely linear model of a vibrating string, so the harmonic implications derived from it ignores non-linearities such as string stiffness, dullness introduced by skin oil and aging, and other material properties of the strings beyond linear density and tension.

The 1-D wave equation is shown below, where T represents the string tension, ρL represents its linear density, and c the wave speed along the string:

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \quad c^2 = \frac{T}{\rho_L} \quad (1)$$

The general complex-form solution to the above differential equation is shown below, where the leading constants are complex-valued:

$$y(x,t) = \underset{\sim}{A} e^{i(\omega t + kx)} + \underset{\sim}{B} e^{i(\omega t - kx)}, \quad k = \frac{\omega}{c} \quad (2)$$

For a guitar string that is fixed at both ends ($x=0$ and $x=L$), we have boundary conditions that restrict the solution space further. When we plug in these boundary conditions and take the real part of the possible solutions, we are left with a solution that can be written as a linear combination of harmonically-related sinusoids.

$$y(0,t) = 0, \quad y(L,t) = 0 \quad \Rightarrow \quad y(x,t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin k_n x \quad k_n = \frac{\pi n}{L} \quad (3)$$

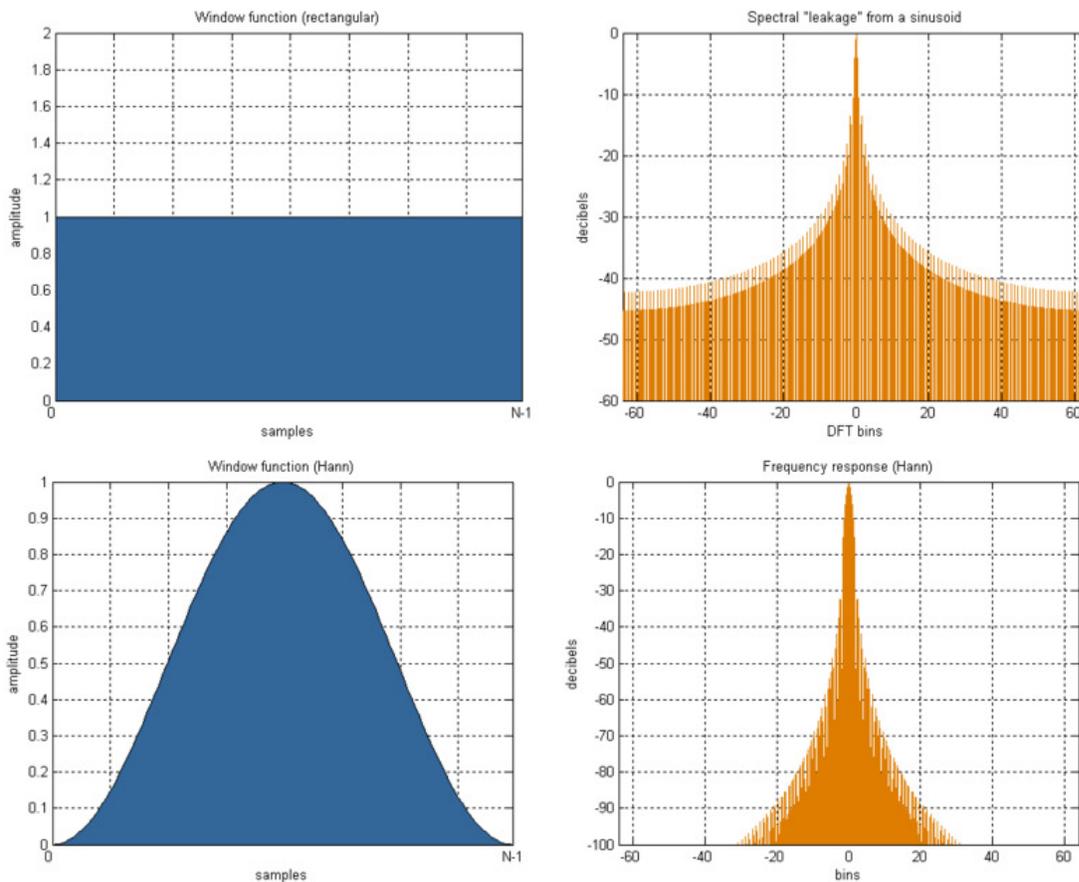
In this case, the fundamental frequency of the string is $\omega_1 = \frac{c \pi n}{L}$ and all other possible frequencies are multiples of this frequency.

We will use these implications of the 1D wave equation as a starting point for our theoretical analysis.

1.2 Experimental Analysis

Our experimental analysis will consist of frequency domain signal analysis. To do this, we use Matlab's *pwelch* function, which implements Welch's method for estimating power spectral density, along with a Hann window of length 32768. Welch's method is a method of spectral analysis that reduces noise introduced by time-dependent variations at the expense of decreased frequency resolution. Welch's method breaks the signal up into overlapping frames of samples and applies an optional window function. The discrete Fourier transform is then applied to each frame. The resulting magnitude spectrum is averaged across frames to get the final spectrum estimate.

The Hann window is a type of *raised cosine window* that is very widely used in signal analysis. Compared to the flat *rectangular window*, the Hann window has a wider main lobe and much smaller side lobe height in its frequency response (see plots below). This means that a Hann window affords much greater suppression of far-reaching frequency component interference while introducing a bit of local distortion in the frequency domain. When examining an audio signal to find the magnitude of its harmonic components, we felt that the Hann window would work well.



We use half-overlapping frames of length 32768 on 3 second monaural audio clips sampled at 44.1kHz. This results in about 8 windowed frames and a frequency resolution of 1.35Hz.

2 Analysis of Picking Location

Anyone who plays guitar or another stringed instrument knows that picking location influences the acoustic qualities of the resulting note. In this section, we attempt to quantify this effect. In addition, we describe our attempt to use the theoretical and experimental analyses to develop a method to determine picking location based on only recorded audio data.

2.1 Picking Location and the 1D Wave Equation

Starting from where we left off in Section 1.1, we can further restrict the solution space of the modes of vibration allowed on a guitar string by imposing zero initial velocity in the string:

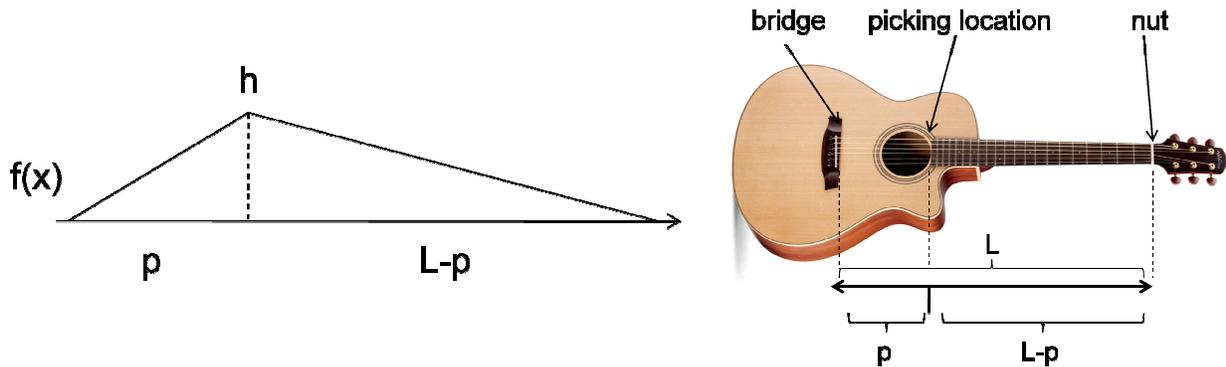
$$\frac{\partial y}{\partial t}(x,0) = 0 \Rightarrow B_n = 0 \Rightarrow y(x,t) = \sum_{n=1}^{\infty} A_n \cos \omega_n t \sin k_n x \quad (4)$$

Then if the initial position of the string is a function $f(x)$, we can solve for the coefficients A_n .

$$y(x,0) = f(x) \Rightarrow A_n = \frac{2}{L} \int_0^L f(x) \sin k_n x dx \quad (5)$$

Note that the solution for $y(x,t)$ in eq. (4) is not time limited and would oscillate forever. In reality, air resistance and internal string friction would result in losses that would cause harmonic components to exponentially decay at different rates. We hope that the short 3 second audio signals will help reduce the influence of these factors on the spectral analysis.

We use a general triangular shape for the initial position function $f(x)$, shown below.



Using equation (5), the amplitude of each harmonic resulting from $f(x)$ above can be written as:

$$A_n = \frac{h}{\pi^2 n^2} \frac{L^2}{p(L-p)} \sin\left(\frac{p\pi n}{L}\right) \quad (6)$$

2.2 Determining Picking Location from an Audio Sample

Since we will already be comparing the theoretical and experimental analyses of the effects of picking location, we thought it fitting to attempt to automatically determine a best fit picking location based only on the data contained in an audio clip of a guitar note being played. The first step in this process is automatically determining the frequency location and amplitude of the harmonic components of a signal.

To do this, we wrote a Matlab script that finds a best fit fundamental frequency and then locates the harmonic components at multiples of this frequency. The steps we use are listed below.

Finding the fundamental (f_0)

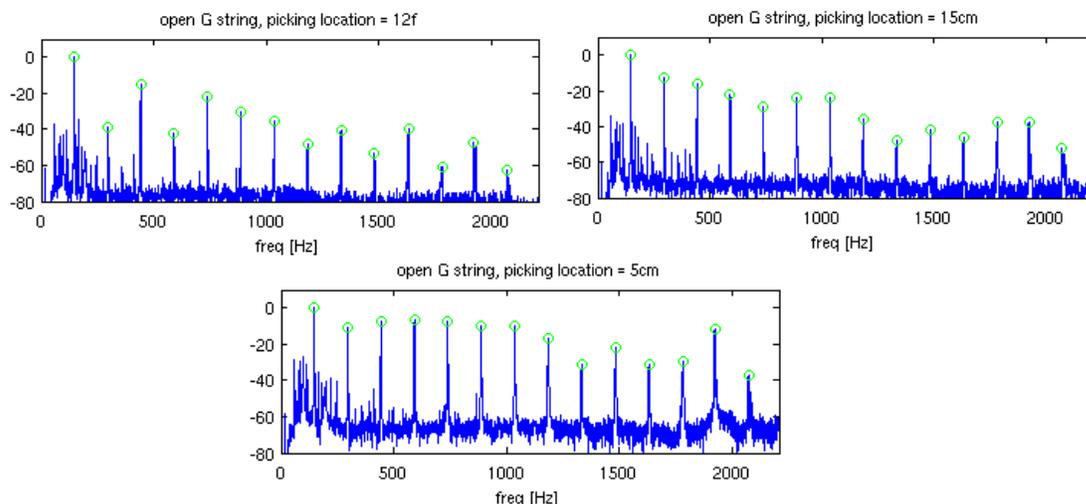
1. Welch's method with Hann window to extract spectrum
2. Convert magnitudes to decibels relative to maximum magnitude
3. Find all local maxima in the spectrum
4. Eliminate local maxima below a string dependent frequency threshold
5. Eliminate local maxima which aren't the largest within a 20Hz window
6. f_0 is set to be the location of the remaining local maximum with the lowest frequency

Finding the harmonics

Steps 1, 2, and 3 as above

4. Eliminate local maxima which are below f_0
5. Eliminate local maxima which aren't the largest within $0.75 \times f_0$

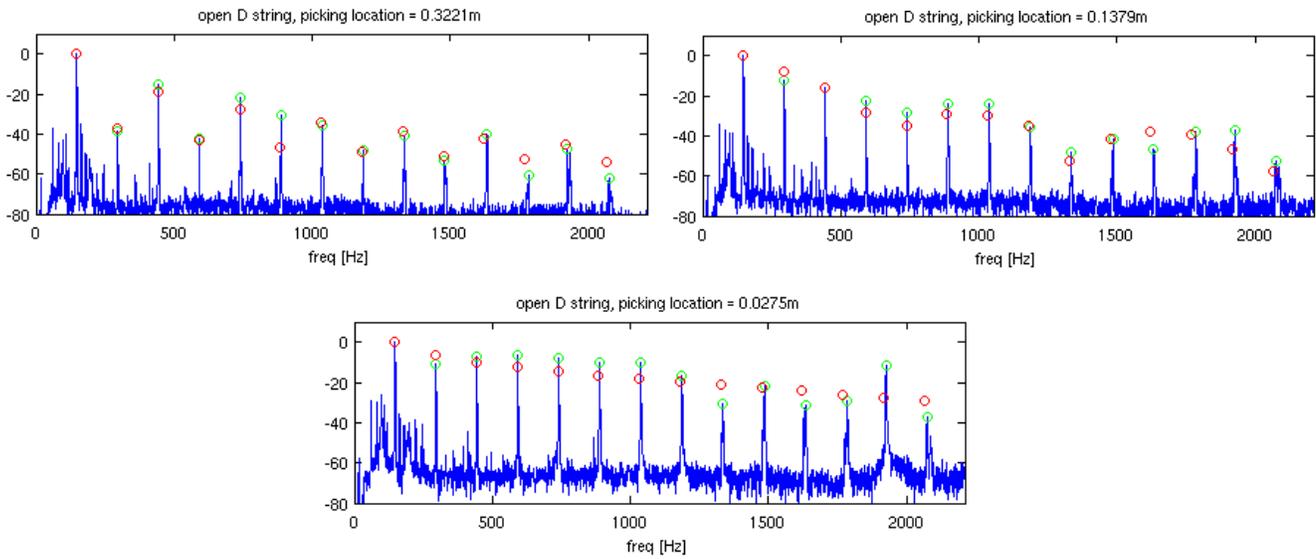
Using these procedures, we locate the first 14 harmonics of each guitar note. The harmonics are denoted by green circles overlaid on the corresponding magnitude spectrum for three different picking locations below – 12th fret (31.65cm from the bridge), 15cm from the bridge, and 5cm from the bridge. As you can see, the method summarized above does a good job of picking out only the harmonics while ignoring all noisy spikes.



Once we have located the first 14 harmonics of a guitar note, we can try to match the actual harmonic distribution with the theoretical harmonic distribution for a given picking distance. To do this, we iterate over the theoretical harmonic distributions for picking locations along the neck of the guitar separated by 0.1mm. Each theoretical distribution is compared to the actual distribution using a

weighted mean-squared error criterion that places more importance on lower harmonics. The weights are computed using a Gaussian curve that decays to 0.5 by the time it reaches the 14th harmonic.

The harmonic distribution with the lowest weighted mean-squared error is chosen as the matching picking location. For the three picking locations shown in the previous plots, the theoretical harmonic distributions chosen by our matching algorithm are shown below as red circles.



The matching picking locations are shown in the table below. Though the matched distances are not exact, they do correlate well with the actual distances used to produce the signals. Differences are most likely caused by non-linearities missing from the model such as string stiffness and the rate of decay of harmonics.

Actual Picking Location	Matched Picking Location
31.65cm (12 th fret)	32.31cm
15cm	13.79cm
5cm	2.75cm

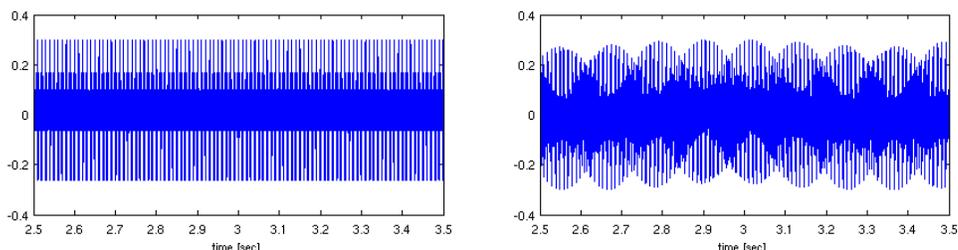
3 Analysis of Tuning Systems

In music theory, a tuning system is the way in which the pitches of musical instruments are related. The acoustic properties of harmonies created by multiple pitches sounding simultaneously are related to the intervals between the pitches. *Just intonation* is a system in which all pitch intervals correspond to a ratio of whole numbers between the frequencies of each pitch. For example, the frequency ratio for a perfect fifth, an interval regarded as consonant or pleasant sounding, is 3:2.

The problem with the just intonation system of tuning is that because pitch frequencies are determined with respect to a single reference note, an instrument would have to be re-tuned every time the musician wants to change keys or chords. To address this problem, western music has widely adopted the *equal temperament* system of tuning. Equal temperament approximates the intervals defined by just intonation using a constant ratio between adjacent frequencies. This ratio is calculated such that there are 11 distinct pitches, or *semitones*, before the next octave is reached (or before the original pitch is doubled in frequency). This ratio is then equal to $\sqrt[12]{2} \approx 1.0595$. Using equal temperament, the ratio for a perfect fifth, an interval of 7 semitones, is $2^{\frac{7}{12}} \approx 1.4983$ as opposed to the 1.5 prescribed by just intonation. This difference, while small, can actually influence the timbre or spectral quality of a musical interval enough that it is unpleasant to a trained ear. A table comparing the two tuning systems for the first 8 musical intervals is shown below.

Interval Name	Exact value in Equal Temperament	Decimal value in Equal Temperament	Just intonation interval	Difference in Hz For a 400Hz root
Unison (C)	$2^{(0/12)}$	1.000000	$1/1= 1.000000$	0
Minor second (C#)	$2^{(1/12)}$	1.059463	$16/15= 1.066667$	2.8816
Major second (D)	$2^{(2/12)}$	1.122462	$9/8= 1.125000$	1.0152
Minor third (D#)	$2^{(3/12)}$	1.189207	$6/5= 1.200000$	4.3172
Major third (E)	$2^{(4/12)}$	1.259921	$5/4= 1.250000$	-3.9684
Perfect fourth (E)	$2^{(5/12)}$	1.334840	$4/3= 1.333333$	-0.6028
Augmented fourth (F#)	$2^{(6/12)}$	1.414214	$7/5= 1.400000$	-5.6856
Perfect fifth (G)	$2^{(7/12)}$	1.498307	$3/2= 1.500000$	-0.6772

The right column of the table shows the frequency difference in Hz between the two tuning systems when the root pitch is set to 400Hz. Differences of a few Hertz are seen for some intervals. These differences have even stronger implications when harmonies consisting of greater than two notes are constructed. For example, a major triad, made up of a root, major third, and perfect fifth, at 400Hz contains pitches with fundamental frequencies 400Hz, 500Hz, and 600Hz using just intonation. Using equal temperament, these frequencies are 400Hz, 504Hz, and 599Hz. Below, signals containing these frequencies are plotted. The left plot contains the just intonation frequencies; the right plot, the equal temperament frequencies. Strong beating is seen in the amplitude envelope which is caused by the frequency deviations in the equal temperament intervals.



Even though just intonation preserves the ideal aspects of musical harmony, it would make sense that acoustic guitars are constructed using equal temperament intervals between the frets. We examine the frequency ratios using the fundamental frequency finding method covered in Section 2.2. The first eight notes on the G string are analyzed and their fundamentals are extracted. The frequency ratios for these notes are listed in the table below along with the corresponding ratio for the two tuning systems.

Interval (Note)	Frequency	Ratio	ET Ratio	JI Ratio	Closer
Unison (G)	199.86	1.000	1.000	1.000	-
Minor 2 nd (G#)	212.64	1.064	1.059	1.067	JI
Major 2 nd (A)	225.43	1.128	1.122	1.125	JI
Minor 3 rd (A#)	238.21	1.192	1.189	1.200	ET
Major 3 rd (B)	252.34	1.263	1.260	1.250	ET
Perfect 4 th (C)	266.47	1.333	1.335	1.333	JI
Augmented 4 th (C#)	282.62	1.414	1.414	1.400	ET
Perfect 5 th (D)	298.77	1.495	1.498	1.500	ET

The rightmost column shows which tuning system explains the observed frequency ratio best. While equal temperament seems to barely edge out just intonation, it is likely that noise in the measurements combined with possibly low-quality construction of the test guitar made the results closer than they should've been. Using equal temperament tuning in a guitar makes much more sense from a practical standpoint.

4 Analysis of Natural Harmonics

In addition to creating new pitches by fretting a guitar, thereby changing the length of the string, new notes can be created by lightly touching a string at a specific location to prevent vibrations at that point. New pitches created in this way are called *natural harmonics*. Doing this adds additional conditions to those imposed by the boundary conditions covered in eq. (3) of Section 1.1. The boundary conditions have already limited the solution space to the following:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos \omega_n t \sin k_n x, \quad k_n = \frac{\pi n}{L} \quad (7)$$

By forcing the displacement on the string to zero at a specific distance from the bridge d , we force the amplitude of the n th harmonic to zero unless $\frac{d}{L}n$ is a whole number, i.e.

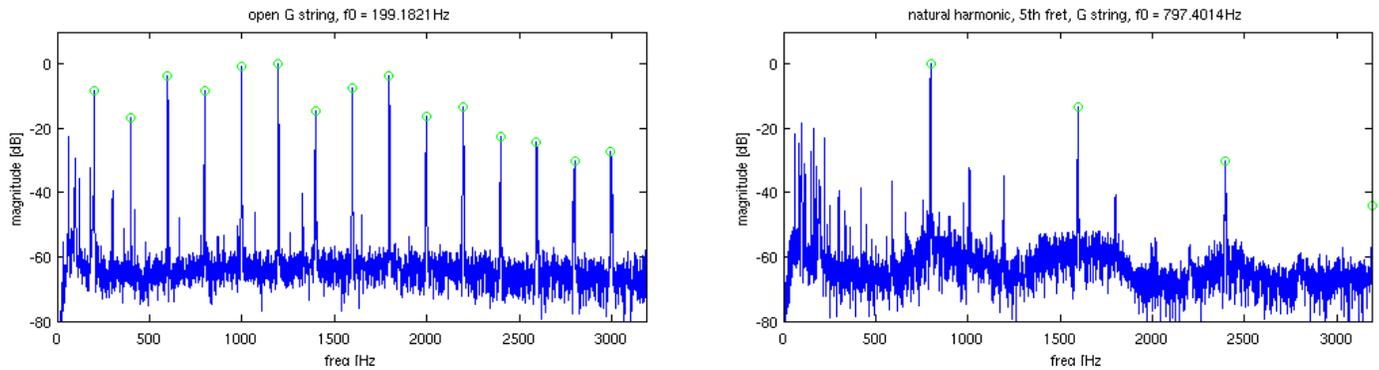
$$y(d,t) = 0 \Rightarrow A_n = 0 \quad \text{unless } \frac{d}{L}n \text{ is a whole number} \quad (8)$$

We approximate the fret locations using the just intonation interval ratios between the open string pitch and the pitch at that fret location. These ratios then represent the ratio of the original length of the string to the fretted length. The chart below shows which of the harmonics of a string are still allowed to vibrate when a finger is placed at the 5th, 7th, and 9th frets. The just intonation ratio for the 5th fret (a perfect 4th) is 4/3. Therefore, the string must be shortened to $\frac{3}{4}$ the original length to get a perfect 4th, and $\frac{d}{L} = \frac{3}{4}$. This ratio is only equal to a whole number when multiplied by a multiple of 4, so we are left with harmonics 4, 8, 12, 16, etc. These harmonics correspond to those of the same pitch as the open string raised by two octaves. Therefore, the natural harmonic created by placing a finger at the 5th fret gives us an interval of 2 octaves.

Likewise, the natural harmonic created at the 7th fret leaves us with harmonics 3, 6, 9, 12, etc. These harmonics correspond to an interval of an octave (x2) plus a perfect fifth (x1.5). A natural harmonic at the 9th fret gives us harmonics at multiples of 5, corresponding to 2 octaves (x4) plus a major third (x1.25). These harmonic distributions can be visualized more easily with the aid of the table below.

Fret (1/JI ratio)	5 (3/4)	7 (2/3)	9 (3/5)
n			
1			
2			
3		♪	
4	♪		
5			♪
6		♪	
7			
8	♪		
9		♪	
10			♪

As an example, we examined the audio signal created by a natural harmonic at the 5th fret on the G string. The spectrum of the original open string is shown on the left with the harmonics labeled with green circles. The 5th fret natural harmonic spectrum appears on the right, and we can see that all harmonics except for the 4th, 8th, 12th, and 16th have been practically eliminated.



5 Discussion

This project has attempted to quantify certain aspects of the acoustic guitar, namely, the influence of picking location on harmonic distribution, the tradeoff between convenience and interval quality present in different tuning systems, and the theoretical and acoustic properties of natural harmonics.

We have shown that the influence of picking location can be modeled and used to recognize the picking location used to play a particular note. Even though a simple linearized model is used, the accuracy is enough to provide useful information. In order to use this method to annotate an automatic guitar music transcription system, the robustness must be increased. This could be accomplished by introducing additional components into the model to describe the decay of harmonics and the influence of the material properties of different strings. In addition, using some sort of supervised (trained) machine learning method to help make decisions about best matching distances would increase the accuracy of the system and make it less sensitive to noise.